

# On Quasiperiodic Space Tilings, Inflation and Dehn Invariants

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## Abstract

We introduce Dehn invariants as a useful tool in the study of the inflation of quasiperiodic space tilings. The tilings by “golden tetrahedra” are considered. We discuss how the Dehn invariants can be applied to the study of inflation properties of the six golden tetrahedra. We also use geometry of the faces of the golden tetrahedra to analyze their inflation properties. We give the inflation rules for decorated Mosseri–Sadoc tiles in the projection class of tilings  $\mathcal{T}^{(MS)}$ . The Dehn invariants of the Mosseri–Sadoc tiles provide two eigenvectors of the inflation matrix with eigenvalues equal to  $\tau = \frac{1+\sqrt{5}}{2}$  and  $-\frac{1}{\tau}$ , and allow to reconstruct the inflation matrix uniquely.

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# 1 Introduction

Existing mathematical models of quasiperiodic tilings of the plane [1, 2, 3] and the 3dimensional space [4, 5, 6, 7, 8] admit an important operation called inflation. Given a tiling of the plane or the space by prototiles  $\{X^i\}$  from a *local isomorphism class of tilings* [3] or *specie* [7], the inflation produces another tiling of the class out of the first one, by blowing up the tiles with a factor  $\lambda$  ( $\lambda$  is bigger then 1 and called the inflation factor) and substituting the  $\lambda$ -scaled tiles  $X^i_{(\lambda)}$  in a particular way by the tiles  $\{X^i\}$  of the original size. Generally, in the process of inflation, the tiles  $X^i$  are cut into pieces (by plane cuts) and these smaller pieces can then be recombined together into the tiles  $X^i_{(\lambda)}$ . The tile  $X^i_{(\lambda)}$  is made out of pieces of tiles  $X^j$  for all  $j$ . Let  $M_j^i$  be the sum of volumes of the pieces of the tiles of the type  $X^j$ . The matrix  $M = M_j^i$  is called the volume inflation matrix.

By its definition, the matrix  $M$  has an eigenvector  $\vec{v}$  with components  $v^i = \text{Vol}(X^i)$ , the volumes of the tiles. The corresponding eigenvalue is  $\lambda^3$ .

In some cases, the matrix  $M$  has rational entries. An example of an exception is the volume inflation matrix of the class of the tilings  $\mathcal{T}^{*(2F)}$  icosahedrally projected from the lattice  $D_6$ , [8, 9], to be discussed later in this paper. (Note: Under the “icosahedral projection” we mean the icosahedrally invariant projection.) Let  $\mathcal{Q}[\lambda]$  be the extension of  $\mathcal{Q}$  by  $\lambda$  and  $G = \text{Gal}(\mathcal{Q}[\lambda]/\mathcal{Q})$  its Galois group. Let  $G\lambda = \{\lambda_1 = \lambda, \lambda_2, \dots, \lambda_k\}$  be the orbit of  $\lambda$ . Then all the  $\lambda_i$  are eigenvalues of the matrix  $M$ .

In many physically interesting cases,  $\lambda$  is a power of the golden mean  $\tau = \frac{1+\sqrt{5}}{2}$ ; the field  $\mathcal{Q}[\lambda]$  is quadratic and therefore volumes can be used to build two eigenvectors (and eigenvalues) of  $M$ .

In this article we address a question of a geometrical meaning of other eigenvectors of  $M$ .

We need several standard definitions. One says that two polyhedra,  $P_1$  and  $P_2$ , are scissor-equivalent (notation:  $P_1 \sim P_2$ ) if  $P_1$  can be cut (by plane cuts) and rebuilt into  $P_2$ .

Assume that there is a function  $\mathcal{F}$  which associates an element of a ring  $\mathcal{K}$  to any polyhedron. The function  $\mathcal{F}$  is called scissor-invariant if  $\mathcal{F}$  enjoys the property:  $P_1 \sim P_2 \Rightarrow \mathcal{F}(P_1) = \mathcal{F}(P_2)$ .

Any scissor-invariant function  $\mathcal{F}$  allows to construct an eigenvector  $\vec{f}$  of the matrix  $M$ ,  $f^i = \mathcal{F}(X^i)$ . The comment about the Galois group holds for the vector  $\vec{f}$  as well.

It is well known that starting from the dimension 3, the space of scissor-invariant functions is nontrivial: in addition to the volume, there are also

Dehn invariants.

In Section 2 of the present article we remind some basic facts about the Dehn invariants.

In Section 3 we consider the Dehn invariants of golden tetrahedra. We use the Dehn invariants as a test of an existence of a stone inflation for the golden tetrahedra (Subsection 3.2). We show that if a rational inflation (that is, an inflation whose inflation matrix has rational entries) for the golden tetrahedra with the inflation factor  $\tau$  exists then the inflation matrix can be uniquely reconstructed with the help of the volumes and the Dehn invariants. This unique inflation matrix  $M_{gt}$  turns out to have non-integer entries which shows that a stone inflation of the golden tetrahedra with the inflation factor  $\tau$  cannot exist. However,  $M_{gt}^3$ , the cube of the matrix  $M_{gt}$ , is integer-valued, so we cannot exclude a possibility of a stone inflation for the golden tetrahedra with the inflation factor  $\tau^3$ .

An alternative proof of the nonexistence of a stone inflation for the golden tetrahedra is given in Subsection 3.3. It is based on the analysis of irrationalities of areas of faces of the golden tetrahedra. The analysis in Subsection 3.3 allows to show that a stone inflation for the golden tetrahedra with the inflation factor  $\tau^k$ ,  $k = 1, 2, 3, \dots$  cannot exist for any  $k$ .

In Subsection 4.1 we present the inflation rules for the *decorated* Mosseri–Sadoc tiles (they are unions of the golden tetrahedra). These rules we obtain by a local derivation from the inflation rules for the *decorated* golden tetrahedra (decoration increases the number of tiles: there are eight decorated golden tetrahedra) as the tiles of the projection class  $\mathcal{T}^{*(2F)}$ , [8, 9, 10].

In Subsection 4.2 we show that the inflation matrix in the case of the Mosseri–Sadoc tilings is uniquely reconstructed from the volumes of the prototiles and their Dehn invariants. Also, we explain in Subsection 4.2 that the inflation matrix for the Mosseri–Sadoc tiles is induced by the inflation matrix for the golden tetrahedra.

For the calculation of the Dehn invariants of the golden tetrahedra we use a Conway–Radin–Sadun theorem (Appendix).

## 2 Dehn invariants

The Dehn invariant of a polyhedron  $P$  takes values in a ring  $\mathbf{R} \otimes \mathbf{R}_\pi$  where  $\mathbf{R}_\pi$  is the additive group of residues of real numbers modulo  $\pi$ ; the tensor product is over  $\mathbf{Z}$ , the ring of rational integers. Denote by  $l_i$  the lengths of edges of  $P$ . Denote by  $\alpha_i$  the corresponding lateral angles and by  $\bar{\alpha}_i$  – the residue classes of  $\alpha_i$  modulo  $\pi$ . The Dehn invariant,  $\mathcal{D}(P)$ , of the polyhedron

$P$  is equal to

$$\mathcal{D}(P) = \sum l_i \otimes \bar{\alpha}_i, \quad (1)$$

with the sum over all edges of  $P$ .

Historically, Dehn invariants appeared in solving the Hilbert's third problem [11] which asks whether one can calculate the volume of a polyhedron without a limiting procedure. More precisely, given two polyhedra of the same volume, can one cut one and paste the pieces to build another one? Or, is equality of volumes of two polyhedra sufficient for their scissor equivalence?

Dehn [12] has shown that the quantity (1) is scissor-invariant and gave an example of two polyhedra of the same volume but having different Dehn invariants. Thus, equality of Dehn invariants is a necessary condition for the scissor equivalence. Later, Sydler [13] has shown that in dimension 3 the equality of volumes and Dehn invariants is also a sufficient condition for the scissor equivalence. See [14] for more information on the Dehn invariants.

### 3 Inflation of golden tetrahedra

In this Section we discuss several aspects of the inflation of the golden tetrahedra, not only the inflation of these tiles as the prototiles *in* the projection class of the tilings  $\mathcal{T}^{*(2F)}$ . The projection class of the locally isomorphic tilings  $\mathcal{T}^{*(2F)}$  and the inflation rules for the tiles in this class have been considered in Refs. [15, 8, 9].

#### 3.1 Golden tetrahedra and their Dehn invariants

Figure 1: (see Fig1.gif) The tiles of the projection class of the tilings  $\mathcal{T}^{*(2F)}$ :  $G^*$ ,  $F^*$ ,  $A^*$ ,  $B^*$ ,  $C^*$  and  $D^*$  (from left to right), the golden tetrahedra. All edges of the tetrahedra are parallel to the 2fold symmetry axes of the icosahedron. They are of the standard length  $\textcircled{2}$  (denoted by 1 in the Figure) and  $\tau\textcircled{2}$  (denoted by  $\tau$  in the Figure),  $\textcircled{2} = \sqrt{\frac{2}{\tau+2}}$ . The representative lateral angles are shown. The  $\mathbf{Z}_3$  rotational symmetry of the tiles  $G^*$  and  $F^*$ , the  $\mathbf{Z}_2$  rotational symmetry of the tiles  $A^*$  and  $B^*$  and reflection symmetry of the tiles  $C^*$  and  $D^*$  allow to reconstruct all other lateral angles.

Golden triangles are triangles with edge lengths 1 and  $\tau$  (in some scale) satisfying the condition: not all edges of a triangle are congruent. There are two golden triangles: with edge lengths  $(1, 1, \tau)$  and with edge lengths  $(1, \tau, \tau)$ . A property of the golden triangles: edges of each of them can be aligned in the plane parallelly to the symmetry axes of a given pentagon.

Golden tetrahedra are tetrahedra with edge lengths 1 and  $\tau$  (therefore the faces of the golden tetrahedra can be either golden or regular triangles) satisfying the condition: not all faces of a tetrahedron are congruent. A property: golden tetrahedra are tetrahedra the edges of which can be aligned in the space parallelly to the 2fold symmetry axes of a given icosahedron.

There could be seven golden tetrahedra but it turns out that one of them is flat. The six non-flat golden tetrahedra,  $G^*$ ,  $F^*$ ,  $A^*$ ,  $B^*$ ,  $C^*$  and  $D^*$  are shown in Fig. 1.

All the lateral angles of the golden tetrahedra are expressed in terms of four acute ( $< \pi/2$ ) angles  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ ,

$$\begin{aligned}\cos \alpha &= \frac{\tau}{\tau + 2} = \frac{1}{\sqrt{5}} , \\ \cos \beta &= \frac{\tau + 1}{\sqrt{3}\sqrt{\tau + 2}} , \\ \cos \gamma &= \frac{\tau + 2}{3\tau} = \frac{\sqrt{5}}{3} , \\ \cos \delta &= \frac{\tau - 1}{\sqrt{3}\sqrt{\tau + 2}} .\end{aligned}\tag{2}$$

In  $\mathbf{R}_\pi$  there are linear dependences between lateral angles  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ .

**Lemma 1.**

$$\alpha + \gamma + 2\beta = \pi ,\tag{3}$$

$$\alpha - \gamma + 2\delta = \pi .\tag{4}$$

*Proof.* Straightforward.  $\square$

Therefore, in  $\mathbf{R}_\pi$  we have relations

$$\bar{\alpha} = -\bar{\beta} - \bar{\delta} ,\tag{5}$$

$$\bar{\gamma} = -\bar{\beta} + \bar{\delta} .\tag{6}$$

Next step is to prove that there are no more relations: in other words, the images of angles  $\beta$  and  $\delta$  are independent in  $\mathbf{R}_\pi$ . Because of (5) and (6) it is sufficient to check the independence of  $\bar{\alpha}$  and  $\bar{\gamma}$ .

**Lemma 2.** The images of angles  $\alpha$  and  $\gamma$  in  $\mathbf{R}_\pi$  are independent.

*Proof.* For notation see Appendix.

The angles  $\alpha$  and  $\gamma$  are pure geodetic. One can check that

$$\alpha = \langle 5 \rangle_1, \quad \gamma = \frac{\pi}{2} - 2\langle 3 \rangle_5. \quad (7)$$

The angles  $\langle 5 \rangle_1$  and  $\langle 3 \rangle_5$  are elements of the basis constructed by Conway–Radin–Sadun. Thus, by the Conway–Radin–Sadun theorem (Appendix), the angles  $\alpha$  and  $\gamma$  are independent.  $\square$

The calculation of Dehn invariants of the golden tetrahedra is now immediate. We shall use  $\beta$  and  $\delta$  as independent angles. We express the Dehn invariants of the golden tetrahedra by the vector  $\vec{d}_{gt}$

$$\vec{d}_{gt} = \mathcal{D} \begin{pmatrix} A^* \\ B^* \\ C^* \\ D^* \\ F^* \\ G^* \end{pmatrix} = \begin{pmatrix} -\tau - 1 \\ \tau + 5 \\ 3\tau - 2 \\ -2\tau \\ -3\tau \\ 3\tau + 3 \end{pmatrix} \otimes \bar{\beta} + \begin{pmatrix} 5\tau - 1 \\ \tau - 1 \\ -2 \\ -2\tau - 3 \\ -3\tau + 3 \\ 3 \end{pmatrix} \otimes \bar{\delta}. \quad (8)$$

The subscript  $gt$  stands for “golden tetrahedra”.

The vector  $\vec{v}_{gt}$  of volumes of the golden tetrahedra is

$$\vec{v}_{gt} = \text{Vol} \begin{pmatrix} A^* \\ B^* \\ C^* \\ D^* \\ F^* \\ G^* \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 2\tau + 1 \\ 1 \\ \tau + 1 \\ \tau \\ \tau + 1 \\ \tau \end{pmatrix}. \quad (9)$$

### 3.2 On inflation of golden tetrahedra

First we show how to use the Dehn invariants as a necessary condition for the existence of the stone inflation. By definition, the inflation is “stone” [7]

if the inflated tiles are composed of the whole original tiles; in other words, one does not need to cut the original tiles into smaller pieces. In particular, it follows that the volume matrix of the stone inflation has integer entries.

**Lemma 1.** The golden tetrahedra as prototiles of a space tiling do not admit a stone inflation with an inflation factor  $\tau$ .

*Proof.* Assume that the stone inflation exists. Let  $M_{gt}$  be its inflation matrix. Since the inflation is stone, the matrix elements of  $M_{gt}$  are rational integers. In particular,  $M_{gt}$  is stable under the action of the Galois group,  $\tau \rightarrow -1/\tau$ .

The vector  $\vec{v}_{gt}$  (the vector of volumes of the tiles, eqn. (9)) is an eigenvector of  $M_{gt}$  with an eigenvalue  $\tau^3$ .

The additivity of Dehn invariants implies that the vector  $\vec{d}_{gt}$  (the vector of Dehn invariants of the tiles, eqn. (8)) is an eigenvector of  $M_{gt}$  with an eigenvalue  $\tau$  (the eigenvalue is  $\tau$  because Dehn invariants have dimension  $[\text{length}]^1$ ). Decomposing the vector of Dehn invariants in  $\bar{\beta}$  and  $\bar{\delta}$  we obtain two eigenvectors of  $M_{gt}$  with the eigenvalue  $\tau$ .

Explicitely, we have for the volume vector:

$$M_{gt} \begin{pmatrix} 2\tau + 1 \\ 1 \\ \tau + 1 \\ \tau \\ \tau + 1 \\ \tau \end{pmatrix} = \begin{pmatrix} 8\tau + 5 \\ 2\tau + 1 \\ 5\tau + 3 \\ 3\tau + 2 \\ 5\tau + 3 \\ 3\tau + 2 \end{pmatrix}, \quad (10)$$

for the  $\bar{\beta}$ -component of the Dehn vector:

$$M_{gt} \begin{pmatrix} -\tau - 1 \\ \tau + 5 \\ 3\tau - 2 \\ -2\tau \\ -3\tau \\ 3\tau + 3 \end{pmatrix} = \begin{pmatrix} -2\tau - 1 \\ 6\tau + 1 \\ \tau + 3 \\ -2\tau - 2 \\ -3\tau - 3 \\ 6\tau + 3 \end{pmatrix}, \quad (11)$$

and for the  $\bar{\delta}$ -component of the Dehn vector:

$$M_{gt} \begin{pmatrix} 5\tau - 1 \\ \tau - 1 \\ -2 \\ -2\tau - 3 \\ -3\tau + 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4\tau + 5 \\ 1 \\ -2\tau \\ -5\tau - 2 \\ -3 \\ 3\tau \end{pmatrix}. \quad (12)$$

The Galois automorphism  $\tau \rightarrow -1/\tau$  produces three more eigenvectors of  $M_{gt}$ . Since the entries of  $M_{gt}$  are integer, to use the Galois automorphism is the same as to decompose vector equalities (10), (11) and (12) in the powers of  $\tau$  (*i.e.* consider  $\tau^0$ - and  $\tau^1$ -components of (10), (11) and (12)). Writing all the columns together we obtain a matrix equality,

$$M_{gt} \begin{pmatrix} 2 & 1 & -1 & -1 & 5 & -1 \\ 0 & 1 & 1 & 5 & 1 & -1 \\ 1 & 1 & 3 & -2 & 0 & -2 \\ 1 & 0 & -2 & 0 & -2 & -3 \\ 1 & 1 & -3 & 0 & -3 & 3 \\ 1 & 0 & 3 & 3 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 5 & -2 & -1 & 4 & 5 \\ 2 & 1 & 6 & 1 & 0 & 1 \\ 5 & 3 & 1 & 3 & -2 & 0 \\ 3 & 2 & -2 & -2 & -5 & -2 \\ 5 & 3 & -3 & -3 & 0 & -3 \\ 3 & 2 & 6 & 3 & 3 & 0 \end{pmatrix}. \quad (13)$$

The matrix  $M_{gt}$  is acting on a  $6 \times 6$  matrix whose first column is  $\tau^1$ -component of (10), the second column is  $\tau^0$ -component of (10); the 3rd and 4th columns are  $\tau^1$ - and  $\tau^0$ -components of (11); the 5th and 6th columns are  $\tau^1$ - and  $\tau^0$ -components of (12).

The eqn. (13) is the matrix equation for the matrix  $M_{gt}$ . We found the complete basis of eigenvectors, therefore the solution is unique and we find

$$M_{gt} = \begin{pmatrix} 2 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1/2 & 1/2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1/2 & 1/2 & 1 & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

The matrix entries of  $M_{gt}$  are not integers therefore a stone inflation with the inflation factor  $\tau$  cannot exist. *Q. E. D.*  $\square$

We actually proved more: we proved that if an inflation with a *rational* inflation matrix existed then the inflation matrix would necessarily be equal to (14). In other words, having assumed that the inflation matrix is rational we could reconstruct it uniquely. This happened because of a coincidence:  $2 \times 3 = 6$ . Here 2 is the order of the Galois group, 3 is the number of independent invariants (the volume and the two Dehn invariants) while 6 is the number of tiles. Due to this coincidence we obtained the matrix equation for  $M_{gt}$  admitting a unique solution. We don't have a good explanation for this coincidence.

An inflation, with the inflation factor  $\tau$  for the golden tetrahedra as the prototiles of the projection class of the tilings  $\mathcal{T}^{*(2F)}$  (obtained by the icosahedrally invariant projection from the  $D_6$  lattice) has been found in



Refs. [8, 9]. There, one has to divide the tiles  $C^*$  and  $G^*$ , each into two subtypes: “blue” and “red”, and these subtypes inflate differently. Therefore, the number of tiles becomes 8. The volume inflation matrix  $M_{\mathcal{T}^*(2F)}$  is equal to

$$\begin{pmatrix} 11\tau - 16 & 0 & 2\tau - 2 & 2\tau - 3 & 0 & 9\tau - 13 & \tau - 1 & 3\tau - 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -2\tau + 4 & 1 & 0 & -\tau + 2 & 1 & -\tau + 3 & 0 & -\tau + 2 \\ -9\tau + 15 & 0 & -2\tau + 4 & -\tau + 2 & 1 & -8\tau + 14 & -\tau + 2 & -2\tau + 4 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ -2\tau + 4 & 1 & 0 & -\tau + 2 & 0 & -\tau + 2 & 0 & -\tau + 2 \\ -9\tau + 15 & 0 & -2\tau + 4 & -\tau + 2 & 0 & -8\tau + 13 & -\tau + 2 & -2\tau + 4 \end{pmatrix} \quad (15)$$

in the following ordering of the tiles:  $A^*$ ,  $B^*$ ,  $C^{*b}$ ,  $C^{*r}$ ,  $D^*$ ,  $F^*$ ,  $G^{*b}$  and  $G^{*r}$ . The upper indices “ $b$ ” and “ $r$ ” denote the “blue” and the “red” variants of tiles, respectively.

It is interesting to note that: 1. for the tiles  $B^*$ ,  $D^*$  and  $F^*$  the inflation matrices  $M_{gt}$  and  $M_{\mathcal{T}^*(2F)}$  give the same results (up to colors); 2. noninteger entries in (14) appear exactly in the columns corresponding to the tiles  $C^*$  and  $G^*$  – the tiles which are getting blue and red colors in the inflation with the matrix (15).

Lemma 1 of this Subsection shows that a stone inflation with the inflation factor  $\tau$  is impossible. We could however try to construct a hypothetical inflation matrix with an inflation factor  $\tau^k$  with integer positive  $k$ ,  $k > 1$ . As in the proof of the Lemma 1, the volume vectors and the vectors of Dehn invariants fix the inflation matrix uniquely: the only possible inflation matrix with the inflation factor  $\tau^k$  can be the matrix  $M_{gt}^k$ . It turns out that there are powers of the matrix  $M_{gt}$  which are integer-valued.

**Lemma 2.** The matrix  $M_{gt}^k$  has integer entries if and only if  $k$  is divisible by 3.

*Proof.* A direct calculation gives

$$M_{gt}^2 = \begin{pmatrix} 7 & 1 & 6 & 3 & 7 & 4 \\ 1 & 1 & 2 & 2 & 1 & 2 \\ 3 & 1 & 5 & 3 & 4 & 3 \\ 3/2 & 1/2 & 3 & 3 & 3 & 1 \\ 7/2 & 1/2 & 4 & 3 & 5 & 2 \\ 2 & 1 & 3 & 1 & 2 & 3 \end{pmatrix} \quad (16)$$

and

$$M_{gt}^3 = \begin{pmatrix} 26 & 5 & 28 & 16 & 30 & 18 \\ 5 & 2 & 8 & 4 & 6 & 6 \\ 14 & 4 & 19 & 12 & 18 & 12 \\ 8 & 2 & 12 & 9 & 12 & 6 \\ 15 & 3 & 18 & 12 & 19 & 10 \\ 9 & 3 & 12 & 6 & 10 & 9 \end{pmatrix}. \quad (17)$$

Thus,  $M_{gt}^3$  is an integer-valued matrix and therefore matrices  $M_{gt}^{3k}$  are integer-valued as well.

It is left to prove that if  $n$  is not a multiple of 3 then  $M_{gt}^n$  is not integer-valued.

By construction, the eigenvalues of  $M_{gt}$  are

$$\tau^3, (-\tau^{-3}), \tau \text{ and } (-\tau^{-1}). \quad (18)$$

Therefore, the minimal polynomial for  $M_{gt}$  is

$$\chi(x) = x^4 - 5x^3 + 2x^2 + 5x + 1, \quad (19)$$

$$\chi(M_{gt}) = 0.$$

A straightforward check shows that if

$$x^4 = 5x^3 - 2x^2 - 5x - 1 \quad (20)$$

then

$$x^n = a_n x^3 + b_n x^2 + c_n x + d_n \quad (21)$$

with

$$a_n = \frac{1}{3} \left( \frac{f_{3(n-1)}}{2} - f_{n-1} \right) \quad (22)$$

and

$$\begin{aligned} b_n &= a_{n+1} - a_n, \\ c_n &= -a_{n+1} + 3a_n + f_n, \\ d_n &= -a_{n+1} + 4a_n + f_{n-1}. \end{aligned} \quad (23)$$

Here  $\{f_n\}$  are Fibonacci numbers defined by:  $f_0 = 0$ ,  $f_1 = 1$  and  $f_{n+1} = f_n + f_{n-1}$ .

Therefore,

$$M_{gt}^n = a_n M_{gt}^3 + b_n M_{gt}^2 + c_n M_{gt} + d_n \mathbf{Id}, \quad (24)$$

where  $\mathbf{Id}$  is the unit matrix.

The numbers  $a_n$ ,  $b_n$ ,  $c_n$  and  $d_n$  are integer. The matrices  $M_{gt}^3$  and  $\mathbf{Id}$  have integer entries. The matrices  $M_{gt}$  and  $M_{gt}^2$  have – at different places – rational entries with the denominator 2. Therefore, the matrix  $M_{gt}^n$  has integer entries if and only if the integers  $b_n$  and  $c_n$  are even which means that

$$a_{n+1} \equiv a_n \pmod{2} \quad (25)$$

and

$$-a_{n+1} + 3a_n + f_n \equiv 0 \pmod{2} . \quad (26)$$

Substitution of (25) into (26) gives  $f_n \equiv 0 \pmod{2}$ . It is well known that  $f_n$  is even if and only if  $n$  is a multiple of 3 (see, e.g., [16], Chapter 6).  $\square$

To conclude: with the help of the Dehn invariants one is able to show that a stone inflation with the inflation factor  $\tau$  is impossible. However one cannot exclude a stone inflation with the inflation factor  $\tau^3$ .

In the next Subsection we shall show, using a different method, that a stone inflation with the inflation factor  $\tau^3$  is impossible as well.

### 3.3 Faces of golden tetrahedra

The Lemma 1 proved in Subsection 3.2 shows that the stone inflation with the inflation factor  $\tau$  is impossible due to the scissor invariants of the tiles – the volumes and the Dehn invariants.

Here we shall give another argument showing the impossibility of a stone inflation. This argument uses the geometry of faces of the tiles.

More precisely, using Dehn invariants amounts to analyzing irrationalities in the lateral angles of the golden tetrahedra. Now we shall analyze irrationalities in the areas of the faces of the golden tetrahedra.

The faces of the golden tetrahedra are golden and regular triangles.

Denote the regular triangle, with the edge length 1, by  $\Delta_r$ , the acute golden triangle (with edge lengths  $\tau$ ,  $\tau$  and 1) by  $\Delta_a$  and the obtuse golden triangle (with edge lengths  $\tau$ , 1 and 1) by  $\Delta_o$ .

For an arbitrary triangle  $\Delta$ , a notation  $\tau^{-k}\Delta$  means the triangle  $\Delta$  scaled by  $\tau^{-k}$ . Also, for a triangle  $\Delta$ , denote a set of triangles  $\{\tau^{-k}\Delta, k = 1, 2, 3, \dots\}$  by  $\tau^{-}\Delta$ .

The areas  $A(\Delta)$  of the triangles are

$$\begin{aligned} A_r &\equiv A(\Delta_r) = \frac{\sqrt{3}}{4} , \\ A_o &\equiv A(\Delta_o) = \frac{\sqrt{\tau+2}}{4} , \\ A_a &\equiv A(\Delta_a) = \frac{\tau\sqrt{\tau+2}}{4} . \end{aligned} \tag{27}$$

The  $\sqrt{\phantom{x}}$  will always denote the positive branch of the square root.

It is interesting to note that the irrationalities in the areas of the faces are exactly the same as in trigonometric functions of the lateral angles (see (2)):  $\sqrt{3}$ ,  $\tau$  and  $\sqrt{\tau+2}$ .

We first prove an intuitively obvious technical Lemma which shows that irrationalities expressing the areas (27) are different.

**Lemma 1.** The irrationalities  $\sqrt{3}$  and  $\sqrt{\tau+2}$  are independent over the field  $\mathbb{Q}[\tau]$ .

*Proof.* The number  $\rho = \sqrt{\tau+2}$  satisfies an equation  $f(\rho) = 0$  where

$$f(x) = x^4 - 5x^2 + 5 . \tag{28}$$

By the Eisenstein criterion (see, e.g., [17], Chapter 3), the polynomial  $f$  is irreducible over  $\mathbb{Q}$ . Moreover,  $f$  splits in  $\mathbb{Q}[\rho]$ : its roots are

$$\pm \sqrt{\tau+2} \text{ and } \pm \sqrt{3-\tau} . \tag{29}$$

The irrationality  $\sqrt{3-\tau}$  belongs to the field  $\mathbb{Q}[\rho]$ : one has

$$\sqrt{3-\tau} = \tau^{-1} \sqrt{\tau+2} \in \mathbb{Q}[\rho] . \tag{30}$$

A splitting field of any polynomial is a Galois extension ([17], Chapter 4). Therefore, the field  $\mathbb{Q}[\rho]$  – as the splitting field of the polynomial  $f$  – is the Galois extension of  $\mathbb{Q}$ .

The automorphism group  $\text{Gal}(\mathbb{Q}[\rho]/\mathbb{Q})$  is isomorphic to the cyclic group  $\mathbb{Z}_4$ , with the generator  $\sigma$ ,

$$\sigma : \sqrt{\tau+2} \rightarrow \sqrt{3-\tau} . \tag{31}$$

In a basis  $1, \sqrt{\tau+2}, \tau$  and  $\sqrt{3-\tau}$  of  $\mathbb{Q}[\rho]$  over  $\mathbb{Q}$ , the action of  $\sigma$  on the other elements of the basis is given by

$$\sigma : \tau \rightarrow -\tau^{-1} \text{ and } \sigma : \sqrt{3-\tau} \rightarrow -\sqrt{\tau+2} . \tag{32}$$

Hence,  $\sigma^4 = 1$ .

By the Fundamental Theorem of Galois Theory (see, e.g., [17], Chapter 4), a quadratic extension of  $\mathcal{Q}$  between  $\mathcal{Q}$  and  $\mathcal{Q}[\rho]$  can be only the fixed field of  $\sigma^2$  which is  $\mathcal{Q}[\tau]$ .

In particular,  $\sqrt{3} \notin \mathcal{Q}[\rho]$  (since, clearly,  $\sqrt{3} \notin \mathcal{Q}[\tau]$ ).  $\square$

*Remark.* It is also easy to prove in an elementary way that the equation  $x^2 = 3$  does not have solutions in  $\mathcal{Q}[\rho]$ .

**Corollary.** The field  $\mathcal{Q}[\rho, \sqrt{3}]$  admits an automorphism  $\phi$  which satisfies:

1.  $\phi : \sqrt{3} \rightarrow -\sqrt{3}$ ;
2. the fixed field of  $\phi$  is  $\mathcal{Q}[\rho]$ .

*Proof.* It follows from the Lemma 1 that the field  $\mathcal{Q}[\rho, \sqrt{3}]$  is a quadratic extension of the field  $\mathcal{Q}[\rho]$ . In characteristic 0, any quadratic extension is Galois ([17], Chapter 4). This immediately implies the existence of the automorphism  $\phi$ .  $\square$

We shall now apply these algebraic preliminaries to the analysis of a stone inflation.

If a stone inflation existed, the faces of inflated tiles would be covered by the faces of the original tiles.

**Lemma 2.** 1. Assume that a regular triangle  $\Delta_r$  is covered by a finite (interior)-disjoint union of regular triangles from  $\tau^- \Delta_r$  and golden triangles from  $\tau^- \Delta_a$  and  $\tau^- \Delta_o$ . Then the golden triangles are absent in the covering. In other words, a regular triangle can be covered by regular triangles only.

2. Similarly, the golden triangles can be covered by the golden triangles only, the regular triangles must be absent in the covering.

*Proof.* Suppose that the triangle  $\Delta_r$  is covered by a finite union of triangles from  $\tau^- \Delta_r$ ,  $\tau^- \Delta_a$  and  $\tau^- \Delta_o$ . Then for the areas we have

$$A_r = p_1(\tau^{-2})A_r + p_2(\tau^{-2})A_a + p_3(\tau^{-2})A_o, \quad (33)$$

where  $p_1$ ,  $p_2$  and  $p_3$  are polynomials with nonnegative integer coefficients and the polynomial  $p_1$  does not have a constant term.

Let  $X = \sqrt{3}(1 - p_1(\tau^{-2}))$  and  $Y = p_2(\tau^{-2})\tau\sqrt{\tau + 2} + p_3(\tau^{-2})\sqrt{\tau + 2}$ . The equality (33) is equivalent to  $X = Y$ .

Applying the automorphism  $\phi$  (Corollary, Lemma 1) to the equality  $X = Y$  we find  $(-X) = Y$  and it follows that  $X = 0$  and  $Y = 0$  separately. Since

each term in the expressions  $p_2(\tau^{-2})A_a$  and  $p_3(\tau^{-2})A_o$  is nonnegative, the equality  $Y = 0$  implies that the polynomials  $p_2(x)$  and  $p_3(x)$  are identically zero. This means that the golden triangles are absent.

The considerations with coverings of the golden triangles are analogous.

□

To prove the nonexistence of a stone inflation we shall consider coverings of the regular triangle.

We shall prove that a regular triangle with the edge of length  $\tau^k$  cannot be covered by regular triangles with the edge lengths  $\tau^i$ ,  $i = 0, \dots, k-1$ . This will imply that there is no stone inflation with the inflation factor  $\tau^k$  for any  $k$ .

In fact, the same arguments can be applied to coverings of any triangle  $\Delta$  by  $\tau^k$ -smaller copies of the same triangle.

Consider an arbitrary triangle  $\Delta$ . Suppose that the triangle  $\tau^k\Delta$  is divided into a finite (interior)-disjoint union of triangles  $\tau^i\Delta$  with  $i = 0, \dots, k-1$ . Consider such division with a smallest possible  $k$ . Then a triangle  $\Delta = \tau^0\Delta$  is necessarily present – otherwise, rescaling by  $1/\tau$  we would obtain the division of the triangle  $\tau^{k-1}\Delta$  contradicting to the minimality of  $k$ .

Denote by  $\alpha_i$  the number of triangles  $\tau^i\Delta$ . We have  $\alpha_i \geq 0$  for  $i = 1, \dots, k-1$  and  $\alpha_0 > 0$ . Put  $\sigma = \tau^2$ . From the area consideration it follows that

$$\sigma^k = \alpha_{k-1}\sigma^{k-1} + \dots + \alpha_0 . \quad (34)$$

It is this statement which will lead to a contradiction.

**Lemma 3.** The number  $\sigma$  cannot satisfy an equation

$$\sigma^k - \alpha_{k-1}\sigma^{k-1} - \dots - \alpha_0 = 0 , \quad (35)$$

where  $\alpha_1, \dots, \alpha_{k-1}$  are nonnegative integer numbers and  $\alpha_0$  is a positive integer number.

*Proof.* The minimal equation (over  $\mathbf{Z}$ ) for  $\sigma = \frac{3+\sqrt{5}}{2}$  is

$$\sigma^2 - 3\sigma + 1 = 0 . \quad (36)$$

Let  $p(x) = x^k - \alpha_{k-1}x^{k-1} - \dots - \alpha_0$ . Assume that  $p(\sigma) = 0$ . This means that one can divide  $p(x)$  by  $x^2 - 3x + 1$ :

$$p(x) = (x^{k-2} + \beta_{k-3}x^{k-3} + \dots + \beta_0)(x^2 - 3x + 1) . \quad (37)$$

Collecting coefficients in powers of  $x$  we obtain the following system:

$$\begin{aligned}
-\alpha_{k-1} &= -3 + \beta_{k-3} \\
-\alpha_{k-2} &= 1 - 3\beta_{k-3} + \beta_{k-4} \\
-\alpha_{k-3} &= \beta_{k-3} - 3\beta_{k-4} + \beta_{k-5} \\
&\vdots \\
-\alpha_2 &= \beta_2 - 3\beta_1 + \beta_0 \\
-\alpha_1 &= \beta_1 - 3\beta_0 \\
-\alpha_0 &= \beta_0
\end{aligned} \tag{38}$$

Let  $\psi_n = f_{2n+2}$  where  $f_n$  are Fibonacci numbers. Then we have  $\psi_0 = 1$ ,  $\psi_1 = 3$  and

$$\psi_{n+1} = 3\psi_n - \psi_{n-1} . \tag{39}$$

Let  $S = \alpha_{k-1}\psi_0 + \alpha_{k-2}\psi_1 + \dots + \alpha_0\psi_{k-1}$ .

Substituting expressions for  $\alpha_i$  from (38) one finds that due to (39) the terms with  $\psi_i$  for  $i > 1$  cancel and one is left with

$$S = -3\psi_0 + \psi_1 \equiv -3 + 3 = 0 , \tag{40}$$

which is impossible since all  $\psi_i$  are positive,  $\alpha_i$  are nonnegative and  $\alpha_0$  is positive.  $\square$

As we have seen, Lemma 3 implies the following statement.

**Corollary.** A regular triangle cannot be covered by  $\tau^k$ -smaller regular triangles.

With these preliminaries we are now prepared to show that a stone inflation for the golden tetrahedra is impossible.

**Proposition.** For the golden tetrahedra, a stone inflation with the inflation factor  $\tau^k$ , with an arbitrary positive integer  $k$ , does not exist.

*Proof.* As it was said above, an existence of a stone inflation implies that the faces of the inflated tiles can be covered by the faces of the tiles of the original size.

In particular, a face which is an inflated regular triangle, would be covered by regular and golden triangles.

Lemma 2 shows that the golden triangles cannot appear in such covering. Therefore, the inflated regular triangle can be covered by regular triangles only – which is impossible by Corollary, Lemma 3.

This contradiction shows that a stone inflation does not exist.  $\square$

*Remark.* The known tilings  $\mathcal{T}^{*(2F)}$  have the following property. The golden tetrahedra in the tiling of the space have their edges parallel to the 2fold symmetry axes of the icosahedron (“the long range orientational order”). The faces of the tiles which are regular triangles are all located in the planes perpendicular to the 3fold symmetry axes of the icosahedron. However the golden triangles are all perpendicular to the 5fold symmetry axes. Therefore if a stone inflation for the tilings  $\mathcal{T}^{*(2F)}$  existed, the regular triangles could be covered only by the smaller regular triangles due to the orientation of the faces. In this case we don’t need Lemmas 1 and 2.

We stress again that an existence of the “rational” inflation rules for the golden tetrahedra (eqn. 14) is hypothetic because in our algebraic approach we do not impose any restriction on the orientations of the tiles in the tiling of the 3dimensional space.

The logic used in this Subsection gives an additional motivation to consider minimal packages of the golden tetrahedra in which the regular faces are all hidden (see Section 4).

## 4 Mosseri–Sadoc tiles

Figure 2: (see Fig2.gif) The outer shape of the “window”  $W(= V_{\perp})$  of the projection class of the tilings  $\mathcal{T}^{(MS)}$  in  $\mathbb{IE}_{\perp}$ . It is the triacontahedron with an edge length ⑤  $= 1/\sqrt{2}$ , the standard length parallel to the 5fold symmetry axes of the icosahedron. The Figure shows the tiles  $a$ ,  $r$ ,  $m$ ,  $s$  and  $z$  in  $\mathbb{IE}_{\parallel}$ . The symmetries of the tiles and the representative lengths of edges are marked. In this paper the standard length ②  $= \sqrt{\frac{2}{\tau+2}}$  is set to 1.

The five prototiles,  $a$ ,  $m$ ,  $r$ ,  $z$  and  $s$  of the projection class of the tilings  $\mathcal{T}^{(MS)}$  (see [18]) are shown in Fig. 2. The tiles  $r$  and  $m$  appear in  $\mathcal{T}^{(MS)}$  always together as a tile  $h$ ,  $h = r \cup m$ , see Fig. 3. The prototiles  $z$ ,  $h$ ,  $s$  and  $a$  are of



Figure 3: (see Fig3.gif) The tiles  $r$  and  $m$  appear in the projection class of the tilings  $\mathcal{T}^{(MS)}$  always together as the union  $h$ ,  $h = r \cup m$ .

Figure 4: (see Fig4.gif) The tiles  $a$ ,  $m$ ,  $r$ ,  $z$  and  $s$  are obtained by the packing of the golden tetrahedra.

the same shape as the prototiles of the inflation class of the tilings introduced by Sadoc and Mosseri [6], and we call them the Mosseri–Sadoc tiles. The tiles  $a$ ,  $m$ ,  $r$ ,  $z$  and  $s$  are composed of the golden tetrahedra [6, 18], as shown in Fig. 4, in such a way that the regular triangles of the golden tetrahedra are all hidden [18]. Hence, the faces of the composed tiles  $a$ ,  $m$ ,  $r$ ,  $z$  and  $s$  are golden triangles only. The same is true for the Mosseri–Sadoc tiles  $z$ ,  $h$ ,  $s$  and  $a$ .

Using additivity of Dehn invariants one finds the vector of Dehn invariants for the Mosseri–Sadoc tiles:

$$\vec{d}_{MS} = \mathcal{D} \begin{pmatrix} z \\ h \\ s \\ a \end{pmatrix} = -5 \begin{pmatrix} \tau \\ 2 \\ \tau - 1 \\ -\tau \end{pmatrix} \otimes \bar{\alpha} . \quad (41)$$

Thus, the space of Dehn invariants for the Mosseri–Sadoc tiles becomes 1-dimensional, only the combination  $\bar{\alpha} = -\bar{\beta} - \bar{\delta}$  appears.

For the vector of volumes for the Mosseri–Sadoc tiles one obtains

$$\vec{v}_{MS} = \text{Vol} \begin{pmatrix} z \\ h \\ s \\ a \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 4\tau + 2 \\ 6\tau + 4 \\ 4\tau + 3 \\ 2\tau + 1 \end{pmatrix} . \quad (42)$$

*Note.* The Mosseri–Sadoc tile  $h$  is the union of the tiles  $m$  and  $r$  introduced in Ref. [18]. The volumes and the Dehn invariants of the tiles  $m$  and  $r$  are

$$\text{Vol}(m) = \frac{1}{12}(2\tau + 3) , \quad \text{Vol}(r) = \frac{1}{12}(4\tau + 1) . \quad (43)$$

$$\mathcal{D}(m) = 5(\tau - 1) \otimes \bar{\alpha} , \quad \mathcal{D}(r) = -5(\tau + 1) \otimes \bar{\alpha} . \quad (44)$$

The Dehn invariants of both of them contain only the combination  $\bar{\alpha}$ . Thus, were the tiles  $m$  and  $r$  not always glued together, we wouldn't be able to

write a matrix equation for the inflation of 5 tiles  $z$ ,  $m$ ,  $r$ ,  $s$  and  $a$ . That the tiles  $m$  and  $r$  in the projection class of the tilings  $\mathcal{T}^{(MS)}$  do appear always together as the prototile  $h$  has been shown in Ref. [18] by the arguments of the projection method expressed in the “orthogonal space”. For the overview of the space tilings obtained by the projection method see Ref. [19].

#### 4.1 Inflation of decorated Mosseri–Sadoc tiles

Figure 5: (see Fig5.gif) The inflation rule for the decorated tile  $a$ :  $\tau a = a \cup s \cup a$ . The “white” arrow marks the edge  $\tau^2 \textcircled{2}$ , the “long” edge in the  $\tau \mathcal{T}^{*(2F)}$ –class of the tilings (the  $\mathcal{T}^{*(2F)}$ –class of the tilings scaled by  $\tau$ ).

Figure 6: (see Fig6.gif) The inflation rule for the decorated tile  $m$ :  $\tau m = a \cup s \cup z \cup a$ . The white arrow is marking the “long” edge in the  $\tau \mathcal{T}^{*(2F)}$ –class of tilings.

Figure 7: (see Fig7.gif) The inflation rule for the decorated tile  $r$ :  $\tau r = z \cup s \cup m \cup r$ .

Mosseri and Sadoc have given the inflation rules for their  $z$ ,  $h$ ,  $s$  and  $a$  tiles [6]. These rules were for the stone inflation [7] of the tiles. The inflation factor is  $\tau = \frac{1+\sqrt{5}}{2}$ . The inflation matrix of the stone inflation of the tiles is the matrix with integer coefficients. It has been given by Sadoc and Mosseri [6]

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad (45)$$

in the following ordering of the tiles:  $z$ ,  $h$ ,  $s$  and  $a$ .

In the case of the Mosseri–Sadoc tiles, the stone inflation is breaking the symmetry of the tiles. The authors of [6] haven’t given a decoration of the tiles which would take care about the symmetry breaking and *uniquely* define the inflation–deflation procedure at *every step*. In [18] it has been shown that the projection class of the locally isomorphic tilings  $\mathcal{T}^{*(2F)}$  (see [15]) can be

locally transformed into the tilings  $\mathcal{T}^{(MS)}, \mathcal{T}^{*(2F)} \longrightarrow \mathcal{T}^{(MS)}$ . The class  $\mathcal{T}^{(MS)}$  of the locally isomorphic tilings of the space by the Mosseri–Sadoc tiles has been defined by the icosahedral projection from the  $D_6$ –lattice [18]. The important property is that *minimal* packages of the six golden tetrahedra in  $\mathcal{T}^{*(2F)}$ , satisfying the condition that their equilateral faces (orthogonal to the 3fold directions) are covered, lead to five tiles  $a, s, z, r$  and  $m$  [18]. Moreover, the tiles  $r$  and  $m$  appear always as the union  $r \cup m$ , that is, the tile  $h$  of Sadoc and Mosseri with three mirror symmetries [18]. See Figs. 2, 3 and 4.

Figure 8: (see Fig8.gif) The inflation rule for the decorated tile  $z$ :  $\tau z = \tau r \cup a$ . The white arrows are marking the “short” and the “long” edges in the  $\tau\mathcal{T}^{*(2F)}$ –class of tilings.

Figure 9: (see Fig9.gif) The inflation rule for the decorated tile  $s$ :  $\tau s = \tau z \cup a$ . The white arrow is marking the “long” edge in the  $\tau\mathcal{T}^{*(2F)}$ –class of tilings.

It is apriori not evident that the inflation rules for the Mosseri–Sadoc tiles in the projection class of the tilings  $\mathcal{T}^{(MS)}$  are the *same* as those suggested by Sadoc and Mosseri [6].

The inflation rules for the  $\mathcal{T}^{*(2F)}$ –tiles *in* the projection class of the tilings  $\mathcal{T}^{*(2F)}$  have been obtained in Refs. [8, 9]. The inflation rules for the prototiles in a projection class of tilings are determined in the orthogonal space by a procedure explained in Refs. [3, 8]. All edges of the  $\mathcal{T}^{*(2F)}$ –tiles are carrying the arrows and some of these arrows are uniquely defining the inflation rules for the  $\mathcal{T}^{*(2F)}$ –tiles [8]. By a local derivation of  $\mathcal{T}^{(MS)}$  from  $\mathcal{T}^{*(2F)}$ , the Mosseri–Sadoc tiles inherit these arrows [10]. The arrows which break the symmetry of  $\mathcal{T}^{(MS)}$ –tiles are defining the inflation procedure uniquely. The inflation–deflation rules for the decorated  $a, m, r, z$  and  $s$  tiles *in* the projection class of the tilings  $\mathcal{T}^{(MS)}$  are obtained through the local derivation from the inflation–deflation rules for the *decorated* golden tetrahedra (eight prototiles!) as the tiles of the projection class  $\mathcal{T}^{*(2F)}$ . We give the inflation rules for  $a, m, r, z$  and  $s$  tiles in Figs. 5 to 9. If we keep in mind that the tiles  $m$  and  $r$  appear in  $\mathcal{T}^{(MS)}$  together as  $h$ ,  $m \cup r = h$ , these are the inflation–deflation rules for the projection class of the tilings  $\mathcal{T}^{(MS)}$  of the space by the decorated Mosseri–Sadoc tiles  $z, h, s$  and  $a$ . We see that the inflation rules for  $\mathcal{T}^{(MS)}$  as a *projection specie* [18] are the *same* (up to the decoration) as for the *inflation specie* given by Mosseri and Sadoc [6]. By the

fact that only the *decorated* Mosseri–Sadoc tiles *do* have the uniquely defined inflation–deflation procedure and by the fact that the inflation rules for the projection and inflation species are the same, we identify the inflation [6] and the projection species [18] and denote them by the same symbol,  $\mathcal{T}^{(MS)}$ .

## 4.2 Dehn invariants and stone inflation of Mosseri–Sadoc tiles

In this Section we show that the inflation matrix for the Mosseri–Sadoc tiles,  $z$ ,  $h$ ,  $s$  and  $a$ , can be uniquely reconstructed from the Dehn invariants (and the volume).

Denote the inflation matrix by  $M_{MS}$ .

The vectors  $\vec{d}_{MS}$  and  $\vec{v}_{MS}$  (see eqns. (41) and (42)) are eigenvectors of the inflation matrix, with the eigenvalues  $\tau$  and  $\tau^3$  correspondingly (we remind that the eigenvalue is equal to the inflation factor to the power which is the dimension of the corresponding invariant).

Explicitely, for the vector of volumes we have

$$M_{MS} \begin{pmatrix} 4\tau + 2 \\ 6\tau + 4 \\ 4\tau + 3 \\ 2\tau + 1 \end{pmatrix} = \begin{pmatrix} 16\tau + 10 \\ 26\tau + 16 \\ 18\tau + 11 \\ 8\tau + 5 \end{pmatrix} \quad (46)$$

and for the the vector of Dehn invariants:

$$M_{MS} \begin{pmatrix} \tau \\ 2 \\ \tau - 1 \\ -\tau \end{pmatrix} = \begin{pmatrix} \tau + 1 \\ 2\tau \\ 1 \\ -\tau - 1 \end{pmatrix}. \quad (47)$$

As for the golden tetrahedra tiles, assume that the inflation matrix is rational. Then, applying the Galois automorphism one finds two more eigenvectors of  $M_{MS}$ . Again, as for tetrahedra, this amounts to the decomposition of (46) and (47) in powers of  $\tau$ .

Together, the four vector equations imply a matrix equation

$$M_{MS} \begin{pmatrix} 4 & 2 & 1 & 0 \\ 6 & 4 & 0 & 2 \\ 4 & 3 & 1 & -1 \\ 2 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 16 & 10 & 1 & 1 \\ 26 & 16 & 2 & 0 \\ 18 & 11 & 0 & 1 \\ 8 & 5 & -1 & -1 \end{pmatrix}. \quad (48)$$

The solution of this equation is unique and we rediscover the matrix (45).

Note that as for the tetrahedra, the uniqueness happens because of the coincidence: the number of tiles equals to the number of invariants times the order of the Galois group.

*Remarks.* 1. The inflation matrix  $M_{MS}$  for the Mosseri–Sadoc tiles is “induced” by the inflation matrix  $M_{gt}$  for the golden tetrahedra in the following sense.

Denote by  $V_{gt}$  a six-dimensional vector space with a basis

$$\{e_{A^*}, e_{B^*}, e_{C^*}, e_{D^*}, e_{F^*}, e_{G^*}\} \quad (49)$$

labeled by the golden tetrahedra. The matrix  $M_{gt}$  acts in the vector space  $V_{gt}$  in an obvious way. We shall denote the corresponding operator by the same symbol  $M_{gt}$ . The lattice  $L_{gt}$  generated by the basis vectors is not preserved by the operator  $M_{gt}$  since the entries of  $M_{gt}$  are not integers.

Denote by  $V_{MS}$  a four-dimensional vector space with a basis

$$\{e_z, e_h, e_s, e_a\} \quad (50)$$

labeled by the Mosseri–Sadoc tiles. The basis vectors generate a lattice  $L_{MS}$ .

A map  $\psi_{gt} : V_{MS} \rightarrow V_{gt}$  given by

$$\begin{aligned} \psi_{gt}(e_z) &= e_{A^*} + e_{C^*} + e_{G^*} , \\ \psi_{gt}(e_h) &= e_{A^*} + e_{B^*} + 2e_{F^*} + 2e_{G^*} , \\ \psi_{gt}(e_s) &= e_{A^*} + 2e_{C^*} , \\ \psi_{gt}(e_a) &= e_{D^*} + e_{F^*} \end{aligned} \quad (51)$$

is an embedding. It is compatible with the lattice structure.

The map  $\psi_{gt}$  reflects the way of packing the golden tetrahedra into the Mosseri–Sadoc tiles (see Fig. 4).

A direct inspection shows that the four-dimensional subspace  $\text{Im}(\psi_{gt})$  of  $V_{gt}$  is invariant under the action of  $M_{gt}$  and the matrix of the induced operator in  $V_{MS}$ , written in the basis (50), coincides with  $M_{MS}$ .

This is quite natural since both matrices,  $M_{gt}$  and  $M_{MS}$ , are uniquely determined by the geometrical data – the volumes and the Dehn invariants.

2. The space  $V_{MS}$  is a subspace in a five-dimensional space  $V'_{MS}$  with a basis

$$\{e_z, e_m, e_r, e_s, e_a\} . \quad (52)$$

The element  $e_h$  is expressed as  $e_h = e_m + e_r$ .

The space  $V'_{MS}$  also maps into  $V_{gt}$ , the second line in (51) gets replaced by

$$\begin{aligned}\psi_{gt}(e_m) &= e_{B^*} + 2e_{F^*} , \\ \psi_{gt}(e_r) &= e_{A^*} + 2e_{G^*} .\end{aligned}\tag{53}$$

It is not an embedding any more:

$$\psi_{gt}(e_r + e_s) = \psi_{gt}(2e_z) .\tag{54}$$

This explains again (see eqs. (44) and the comment after them) that the inflation matrix for the five tiles  $a$ ,  $m$ ,  $r$ ,  $z$  and  $s$  cannot be reconstructed from the Dehn invariants and the volumes (in other words, from the matrix  $M_{gt}$ ).

In fact, the inflation matrix for the tiles  $a$ ,  $m$ ,  $r$ ,  $z$  and  $s$  which reads (in this ordering of the tiles)

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}\tag{55}$$

is degenerate, so it cannot be induced by the nondegenerate matrix  $M_{gt}$ .

3. Denote by  $V_{\mathcal{T}^*(2F)}$  an eight-dimensional vector space with a basis

$$\{\tilde{e}_{A^*}, \tilde{e}_{B^*}, \tilde{e}_{C^{*b}}, \tilde{e}_{C^{*r}}, \tilde{e}_{D^*}, \tilde{e}_{F^*}, \tilde{e}_{G^{*b}}, \tilde{e}_{G^{*r}}\}\tag{56}$$

labeled by the coloured golden tetrahedra. The matrix  $M_{\mathcal{T}^*(2F)}$  becomes an operator acting in the space  $V_{\mathcal{T}^*(2F)}$ .

Define a map  $\psi_{\mathcal{T}^*(2F)} : V_{MS} \rightarrow V_{\mathcal{T}^*(2F)}$  by

$$\begin{aligned}\psi_{\mathcal{T}^*(2F)}(e_z) &= \tilde{e}_{A^*} + \tilde{e}_{C^{*b}} + \tilde{e}_{G^{*r}} , \\ \psi_{\mathcal{T}^*(2F)}(e_h) &= \tilde{e}_{A^*} + \tilde{e}_{B^*} + 2\tilde{e}_{F^*} + \tilde{e}_{G^{*b}} + \tilde{e}_{G^{*r}} , \\ \psi_{\mathcal{T}^*(2F)}(e_s) &= \tilde{e}_{A^*} + \tilde{e}_{C^{*b}} + \tilde{e}_{C^{*r}} , \\ \psi_{\mathcal{T}^*(2F)}(e_a) &= \tilde{e}_{D^*} + \tilde{e}_{F^*} .\end{aligned}\tag{57}$$

The map  $\psi_{\mathcal{T}^*(2F)}$  is an embedding.

Again, one can directly check that the subspace  $\text{Im}(\psi_{\mathcal{T}^*(2F)})$  is invariant under the operator  $M_{\mathcal{T}^*(2F)}$  and the matrix of the induced operator in  $V_{MS}$ , written in the basis (50), coincides with  $M_{MS}$ .

The map  $\psi_{\mathcal{T}^{*(2F)}}$  can be considered as a “colouring” of the map  $\psi_{gt}$ . One can show that this colouring is unique.

4. The map  $\psi_{\mathcal{T}^{*(2F)}}$  also extends to the map from the five-dimensional space  $V'_{MS}$ , the second line in (57) gets replaced by

$$\begin{aligned}\psi_{\mathcal{T}^{*(2F)}}(e_m) &= \tilde{e}_{B^*} + 2\tilde{e}_{F^*} , \\ \psi_{\mathcal{T}^{*(2F)}}(e_r) &= \tilde{e}_{A^*} + \tilde{e}_{G^{*b}} + \tilde{e}_{G^{*r}} .\end{aligned}\tag{58}$$

However it is still an embedding.

As we have seen in Subsection 4.1, not only the inflation matrix but the actual inflation for the Mosseri–Sadoc tiles (as well as for the five tiles  $z$ ,  $m$ ,  $r$ ,  $s$  and  $a$ ) is induced by the inflation for  $\mathcal{T}^{*(2F)}$ .

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## Appendix: Geodetic angles

In Section 3.2 we showed that the space of Dehn invariants for the golden tetrahedra is 2dimensional. The proof is based on a theorem of Conway, Radin and Sadun [20]. For completeness we briefly remind the needed results from [20].

*Definition.* An angle  $\theta$  is called “pure geodetic” if  $\sin^2 \theta$  is rational.

Let  $\mathcal{E}$  be a vector space spanned over  $\mathbb{Q}$  by pure geodetic angles. In [20] a basis of the vector space  $\mathcal{E}$  is constructed. It is useful to know the basis: one can check whether some given angles are  $\mathbb{Q}$ –independent.

An element of the basis of  $\mathcal{E}$  is denoted by  $\langle p \rangle_d$ . Here  $p$  is a prime integer. The positive integer  $d$  has to satisfy two conditions:

1.  $d$  is square–free;
2.  $(-d)$  is a square modulo  $p$ .

If  $p = 2$  then  $d \equiv 7 \pmod{8}$  additionally.

To define  $\langle p \rangle_d$  one solves an equation  $4p^s = a^2 + db^2$  for  $a$  and  $b$ , with a smallest positive  $s$ . For  $d = 3$  one requires  $b \equiv 0 \pmod{2}$ ; For  $d = 1$  one requires  $b \equiv 0 \pmod{4}$ .

Now,

$$\langle p \rangle_d = \frac{1}{s} \arccos \frac{a}{2p^{s/2}} . \quad (59)$$

**Theorem** (Conway–Radin–Sadun). The angles  $\langle p \rangle_d$  together with  $\pi$  form a basis in  $\mathcal{E}$ .

## References

- [1] R. Penrose, “The role of aesthetics in pure and applied mathematical research”, *Bull. Inst. Math. Appl.* **10** (1974) 266–271
- [2] B. Grünbaum & G.C. Shepard, *Tilings and Patterns*, San Francisco: W.H. Freeman 1987
- [3] M. Baake, P. Kramer, M. Schlottmann and D. Zeidler, “Planar patterns with fivefold symmetry as sections of periodic structures in 4-space” *Int. J. Mod. Phys.* **B4** (1990) 2217–2267
- [4] P. Kramer, “Non-periodic Central Space Filling with Icosahedral Symmetry using Copies of Seven Elementary Cells”, *Acta Cryst.* **A38** (1982) 257–264
- [5] J. E. S. Socolar and P. J. Steinhardt, “Quasicrystals. II Unit-cell configurations”, *Phys. Rev.* **B34** (1986) 617–647
- [6] R. Mosseri and J. F. Sadoc, “Two and three dimensional non-periodic networks obtained from self-similar tiling”, *The Physics of Quasicrystals*, eds. P. J. Steinhardt and S. Ostlund, World Scientific (1987) 720–734
- [7] L. Danzer, “Quasiperiodicity; local and global aspects”, in *Lecture Notes in Physics* 382, eds. V.V. Dodonov and V.I. Man’ko, Springer 1991, pp. 561–572
- [8] Z. Papadopolos, C. Hohneker and P. Kramer, “Tiles-inflation rules for the canonical tilings  $\mathcal{T}^{*(2F)}$  derived by the projection method”, preprint math-ph/9909012 to be published in the Special Issue of Discrete Mathematics in honor of Ludwig Danzer



- [9] C. Hohneker, P. Kramer, and Z. Papadopolos, “Tiles–inflation for the canonical tiling  $\mathcal{T}^{*(2F)}$ ”, in *GROUP21 Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras, Volume 2*, eds.: H.-D. Doebner, W. Scherer, and C. Schulte, World Scientific, Singapore (1997) 982–986
- [10] Z. Papadopolos and O. Ogievetsky, “On Inflation Rules for Mosseri–Sadoc Tilings”, preprint, to be published in Proc. of the *7th International Conference on Quasicrystals*, Stuttgart, Germany, 20–24 September 1999
- [11] D. Hilbert, “Mathematische Probleme”, *Nachr. Gesellschaft Wiss. Göttingen* (1900)
- [12] M. Dehn, “Über den Rauminhalt”, *Göttingen Nachr. Math. Phys.* (1900) 345–354; *Math. Ann.* **55** (1902) 465–478
- [13] J. -P. Sydler, “Conditions nécessaires et suffisantes pour l’équivalence des polyèdres de l’espace euclidien à trois dimensions”, *Comm. Math. Helv.* **40** (1965) 43–80
- [14] V. G. Boltyanski, *Hilbert’s Third Problem*, Nauka 1977 (in russian)
- [15] P. Kramer, Z. Papadopolos and D. Zeidler, “Symmetries of icosahedral quasicrystals”, in Proc. of the Symp. *Symmetries in Science V: Algebraic structures, their representations, realizations and physical applications*, ed. B. Gruber et al., Plenum Press (1991) 395–427
- [16] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics*, Addison–Wesley, 1989
- [17] A. Clark, *Elements of Abstract Algebra*, Dover Publications, New York, 1971
- [18] P. Kramer and Z. Papadopolos, “The Mosseri–Sadoc tiling derived from the root–lattice  $D_6$ ”, *Can. J. Phys.* **72** (1994) 408–414
- [19] Z. Papadopolos and P. Kramer, “Models of icosahedral quasicrystals from 6D lattice”, *Proceedings of the International Conference on Aperiodic Crystals, Aperiodic ’94*, ed. by G. Chapuis et al., World Scientific, Singapore (1995), pp. 70–76
- [20] J. H. Conway, C. Radin, L. Sadun, “On Angles whose squared trigonometric functions are rational”, preprint math-ph/9812019

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